Chapter 16 - Deflections of Beams
Reading: Chapter 16 — Pages 577-583

16-1
Introduction

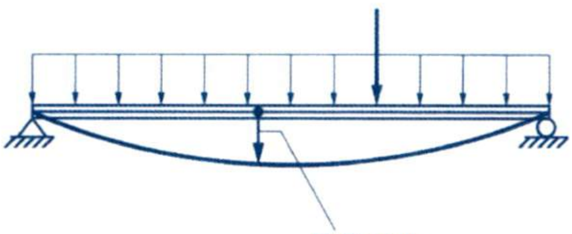
When designing a beam and structures in general, we need to consider **serviceability requirements** as well as its strength requirements.

Serviceability:

- It implies satisfactory performance of the structure under service loads, without discomfort to the user due to excessive deflection, cracking, vibration and so on.
- Other considerations of serviceabilty are durability, comfort, appearance, and maintainability.

The main serviceability requirement is **deflection**.

- When a beam is subjected to transverse loads that produce bending moment, the beam will deflect from its unloaded position.
- The verticle displacement of a point on a horizontal beam is called the *deflection* of the beam.



Deflections are limited to ensure the functional performance of a structure.

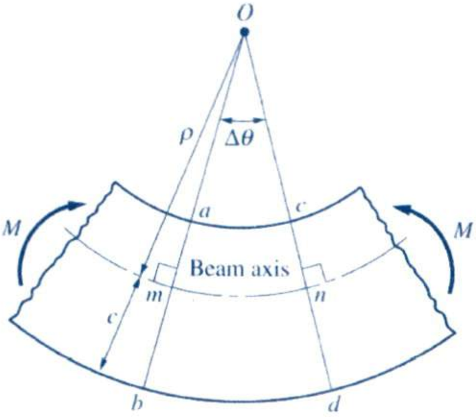
The adverse effect of excessive deflections are:

1. Perceptable vibration of floor systems.
2. Cracking of ceilings and walls and floor finish materials.
3. Poor performance of adjacent building components such as doors and windows.
4. Asthetic view is spoiled.
5. Creates feeling of lack of safety.
6. Creates ponding of water on roof slabs.

Beam deflection values are relatively small compared to overall dimensions of the section and its span length. However, because deflection criterion may control in some instances, the structural designer needs to be able to calculate the anticipated deflection in order to compare this value to the limiting values as set forth by the applicable building code.

The development of beam deflection methods is based on the following assumptions:
1. The beam is homogeneous and obeys Hooke's law, having an equal modulus of elasticity in tension and compression, and the bending is within the elastic range.
2. The beam has a vertical plane of symmetry on which the loads and reactions act.
3. The deflections are small and are caused by bending only. The deflection due to shear is negligible.

16-2
Relationship Between Curvature and Bending Moment



- Consider a beam segment bent into a concave upward curvature due to a positive bending moment.
- Sections ab and cd, remain planar and normal to the axis of the beam.
- The point of intersection O of lines ab and cd is called the *center of curvature*.
- The displacement ρ (the Greek lowercase letter rho) from O to the beam axis is called the *radius of curvature*.

The length of fiber mn along the axis of the beam remains unchanged.
Fiber bd is elongated.

Let the angle ΔΘ at O be measured in radians.
Length mn = $\rho\,\Delta\Theta$
Length bd = $(\rho + c)\,\Delta\Theta$

$$\varepsilon_{max} = \frac{\Delta L}{L} = \frac{bd - mn}{mn} = \frac{(\rho + c)\,\Delta\Theta - \rho\,\Delta\Theta}{\rho\,\Delta\Theta} = \frac{c}{\rho}$$

For elastic bending, $\sigma_{max} = E\varepsilon_{max}$ → $\sigma_{max} = \frac{Ec}{\rho}$

Substituting in the flexure formula $\sigma_{max} = Mc/I$ and solving for ρ we get:

$$\rho = \frac{EI}{M}$$

where,
E is the modulus of elasticity of the material
I is the moment of inertia of the cross-sectional area
M is the bending moment at that section

The radius of curvature ρ of a beam at any section varies inversely with the bending moment at that section.

As the value of bending moment (M)↑ the radius of curvature (ρ) ↓

16-3
The Formula Method

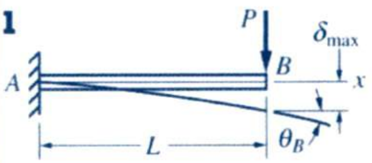
- There are two primary methods for determing beam deflection:

1. Moment-Area Method
2. Formula Method.

For this class we will focus on standard AISC formulas given in Table 16-1.

Typically, designers are concerned with Maximum Deflection (Table 16-1, Column 1).

TABLE 16–1 Beam Deflection Formulas

Beam Loading and Deflection	Maximum Deflection	Slope at End(s)	Deflection Equations
1 	$\delta_{max} = \frac{PL^3}{3EI}$	$\theta_B = \frac{PL^2}{2EI}$	$\delta = \frac{Px^2}{6EI}(3L - x)$

From the formulas derived by integral calculus, we see that deflection and slope are inversely proportional to the product EI, where E is the modulus of elasticity of the material and I is the moment of inertia of the cross-sectional area.

EI is called the flexural rigidity of the beam. It is an indication of the resistance of the beam to deflection. A beam with a greater value of EI is stiffer and will deflect less.

Deflection ↓ as Stiffness ↑

The deflections and slopes computed from the formulas are usually very small.
Units should be in inches (in.) or millimeters (mm).

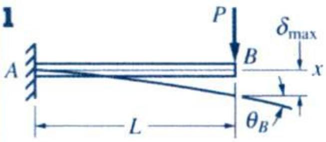
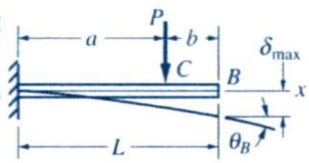
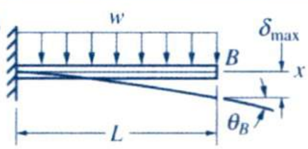
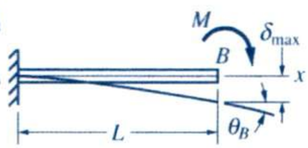
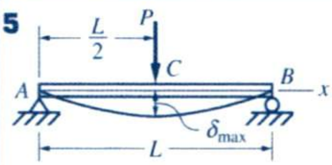
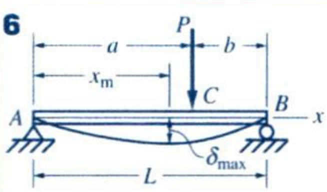
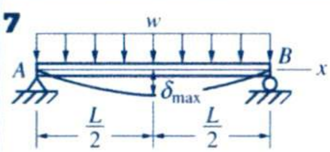
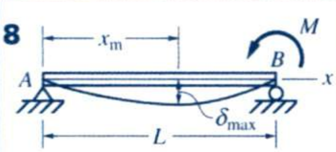
Units must be CONSISTENT and watched carefully.

Uniform Loads
Pounds per inch (lb/in) or Kips per inch (Kip/in)

Span Length
Units that match E

Note: Method of Superposition Applies

TABLE 16–1 Beam Deflection Formulas

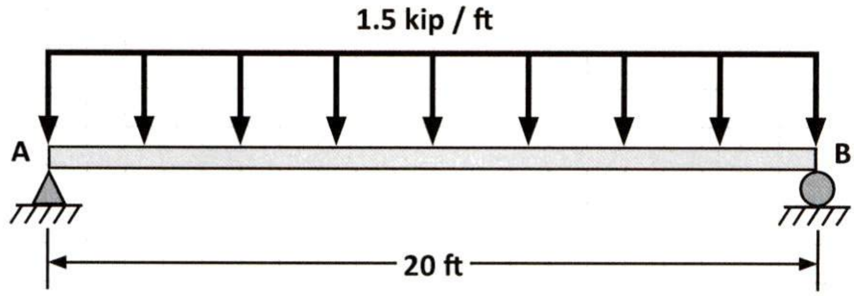
Beam Loading and Deflection	Maximum Deflection	Slope at End(s)	Deflection Equations
1 	$\delta_{max} = \frac{PL^3}{3EI}$	$\theta_B = \frac{PL^2}{2EI}$	$\delta = \frac{Px^2}{6EI}(3L - x)$
2 	$\delta_{max} = \frac{Pa^2}{6EI}(3L - a)$	$\theta_B = \frac{Pa^2}{2EI}$	$\delta_{AC} = \frac{Px^2}{6EI}(3a - x)$ $\delta_{CB} = \frac{Pa^2}{6EI}(3x - a)$
3 	$\delta_{max} = \frac{wL^4}{8EI}$	$\theta_B = \frac{wL^3}{6EI}$	$\delta = \frac{wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
4 	$\delta_{max} = \frac{ML^2}{2EI}$	$\theta_B = \frac{ML}{EI}$	$\delta = \frac{Mx^2}{2EI}$
5 	$\delta_{max} = \frac{PL^3}{48EI}$	$\theta_A = \theta_B = \frac{PL^2}{16EI}$	$\delta_{AC} = \frac{Px}{48EI}(3L^2 - 4x^2)$
6 	For $a > b$: $\delta_{max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2 - b^2}{3}}$	$\theta_A = \frac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = \frac{Pa(L^2 - a^2)}{6EIL}$	$\delta_{AC} = \frac{Pbx}{6EIL}(L^2 - x^2 - b^2)$ $\delta_{CB} = \frac{Pb}{6EIL}\left[\frac{L}{b}(x - a)^3 + (L^2 - b^2)x - x^3\right]$
7 	$\delta_{max} = \frac{5wL^4}{384EI}$	$\theta_A = \theta_B = \frac{wL^3}{24EI}$	$\delta = \frac{wx}{24EI}(L^3 + x^3 - 2Lx^2)$
8 	$\delta_{max} = \frac{ML^2}{9\sqrt{3}EI}$ at $x_m = \frac{L}{\sqrt{3}}$	$\theta_A = \frac{ML}{6EI}$ $\theta_B = \frac{ML}{3EI}$	$\delta = \frac{Mx}{6EIL}(L^2 - x^2)$

Example 1

A W12 X 65 steel section is used in a 20-ft simple span. Compute the maximum deflection due to the uniform load of 1.5 kip/ft applied to the entire span.

$E = 29 \times 10^3$ ksi

Allowable deflection = L/360



W 12 x 65

Solution.

Step 1. Knowns

Determine the Total Load

Live Load (LL) $w = \frac{1.5 \text{ kip}}{\text{ft}} \times \frac{1 \text{ ft}}{12 \text{ in}} = 0.125 \text{ kip/in.}$

Dead Load (DL) (Beam wt) $65 \text{ lb/ft} \times \frac{1 \text{ ft}}{12 \text{ in}} \times \frac{\text{kip}}{1000 \text{ lb}} = 0.0054 \text{ kip/in}$

Total Load = LL + DL = 0.125 kip/in. + 0.0054 kip/in = 0.1304 kip/in.

$L = 20 \text{ ft} \times \frac{12 \text{ in}}{\text{ft}} = 240 \text{ in.}$

$E = 29\,000$ ksi $\qquad I = 533 \text{ in.}^4$ (Table A-1 (a))

$\delta_{allow} = \frac{L}{360}$

Step 2. Table 16-1, case 7

$$\delta_{MAX} = \frac{5wL^4}{384\,EI} = \frac{5(0.1304 \text{ K/in})(240 \text{ in.})^4}{384(29\,000 \text{ ksi})(533 \text{ in.}^4)} = \frac{2{,}163{,}179{,}520 \text{ in.}^3}{5{,}935{,}488{,}000 \text{ in.}^2} = 0.3644 \text{ in.}$$

Step 3:

$$\delta_{allow} = \frac{L}{360} = \frac{240 \text{ in.}}{360} = 0.667 \text{ in.}$$

$\delta_{MAX} = 0.3644 \text{ in.} \leq \delta_{allow} = 0.667 \text{ in.}$ ✓

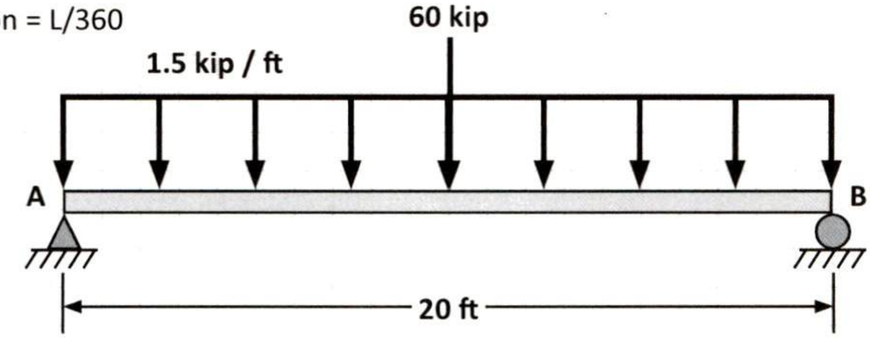
Beam is ok for Deflection

Example 2

A W12 X 65 steel section is used in a 20-ft simple span. Compute the maximum deflection due to the uniform load of 1.5 kip/ft applied to the entire span and a 60 kip concentrated load at midspan.

$E = 29 \times 10^3$ ksi

Allowable deflection = L/360



Solution.

By Superposition,

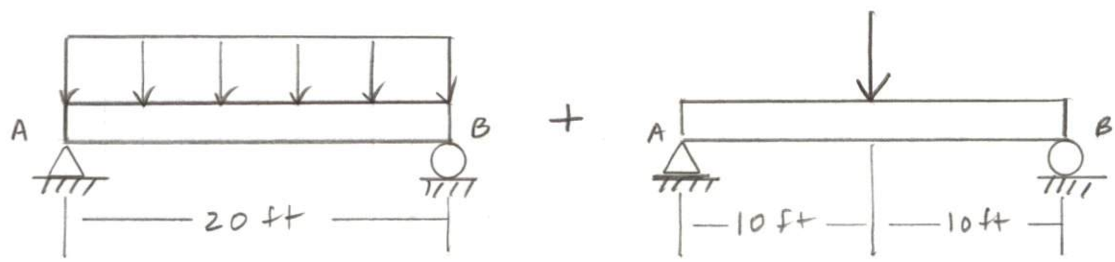


Table 16-1, Case 7 (Example 1)

Table 16-1, Case 5

Recall From EI

$L = 240$ in.

$E = 29\,000$ ksi

$I = 533$ in.^{4}

$$\delta_{MAX} = 0.3644 \text{ in} + \frac{PL^3}{48EI}$$

$$\frac{PL^3}{48EI} = \frac{60 \text{ kip } (240 \text{ in})^3}{48(29\,000 \text{ ksi})(533 \text{ in.}^4)}$$

$$= 1.118 \text{ in.}$$

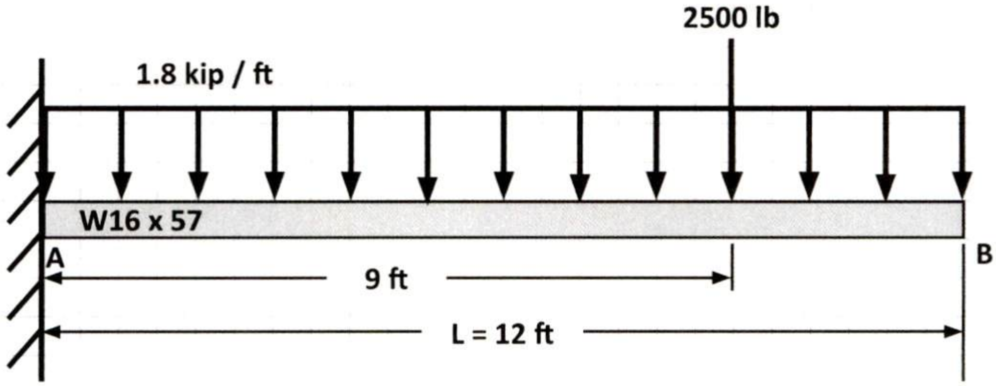
$$\delta_{Total} = 0.3644 \text{ in.} + 1.118 \text{ in} = 1.4824 \text{ in.}$$

$$1.4824 \text{ in} > \delta_{allow} = 0.667 \text{ in}$$

Beam is not Satifactory for Deflection

Example 3

A W16 x 57 steel section is used in a 12-ft cantilever span. Compute the maximum deflection due to a uniform load of 1.8 kip/ft (includes beam weight) and a concentrated load of 2500 lb at 9-ft from the fixed-end A. E = 30 x 10^3 ksi. The maximum allowable deflection is L/360 of the span length.



Solution.

Step1. Knowns

$$P = 2500 \text{ lb} \times \frac{\text{kip}}{1000 \text{ lb}} = 2.5 \text{ kip}$$

$$a = 9 \text{ ft} \times \frac{12 \text{ in}}{\text{ft}} = 108 \text{ in.}$$

$$w = 1.8 \text{ kip/ft} \times \frac{1 \text{ ft}}{12 \text{ in}} = 0.15 \text{ kip/in}$$

$$L = 12 \text{ ft} \times \frac{12 \text{ in}}{\text{ft}} = 144 \text{ in.}$$

Table A-1(a)

W16 X 57

$I = 758 \text{ in}^4$

$$\delta_{allow} = \frac{L}{360} = \frac{144 \text{ in}}{360} = 0.40 \text{ in.}$$

Step 2. Table 16-1, Case 2 and Case 3

Case 2

$$\delta_{MAX} = \frac{Pa^2}{6EI}(3L - a) = \frac{2.5\text{ kip}(108\text{ in})^2}{6(30\,000\text{ ksi})(758\text{ in}^4)}\left(3(144\text{ in}) - 108\text{ in}\right)$$

$$= \frac{29{,}160\text{ in}^2}{136{,}440{,}000\text{ in}^2}(324\text{ in.})$$

$$= 0.06925\text{ in.}$$

Case 3

$$\delta_{MAX} = \frac{wL^4}{8EI} = \frac{0.15\text{ kip/in.}\,(144\text{ in})^4}{8(30\,000\text{ ksi})(758\text{ in}^4)} = \frac{64{,}497{,}254.4\text{ in}^3}{181{,}920{,}000\text{ in}^2}$$

$$= 0.3545\text{ in.}$$

Total Deflection

$$\delta_{MAX} = 0.06925\text{ in} + 0.3545\text{ in} = 0.4238\text{ in.}$$

Step 3.

$$\delta_{Allow} = \frac{L}{360} = \frac{144\text{ in}}{360} = 0.4\text{ in.}$$

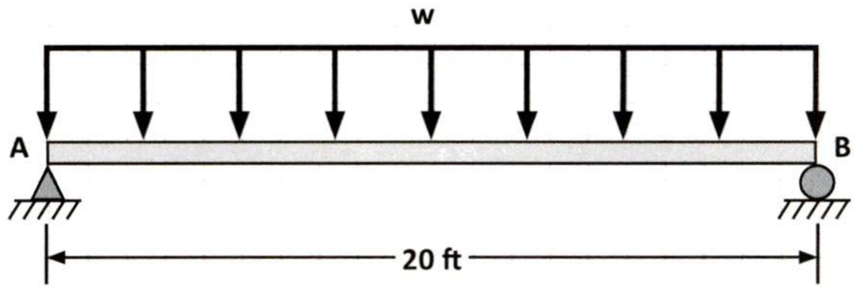
$$\delta_{MAX} = 0.4238\text{ in} > \delta_{allow} = 0.4\text{ in}$$

Beam is not satisfactory in Deflection

Example 4

A W21 x 62 steel section is used in a 20-ft simple span. Determine the maximum allowable uniform load w that the beam can carry if the allowable flexural stress is 24 ksi, the allowable shear stress is 14.5 ksi, and the allowable deflection is L/360 of the span length. E = 30 x 10^3 ksi

$E = 30{,}000$ ksi



Solution.

Step 1.

W21×62 [Table A-1(a)]

$I = 1330\ in.^4$

$S = 127\ in^3$

$d = 20.99$ in.

$t_w = 0.4$ in.

$\sigma_{allow} = 24$ ksi

$\tau_{allow} = 14.5$ ksi

$$\delta_{allow} = \frac{L}{360} = \frac{240\ in}{360} = 0.667\ in$$

$$L = 20\ ft \times \frac{12\ in}{ft} = 240\ in.$$

Step 2. Flexural Stress

Bending Moment

$$M_{allow} = S\,\sigma_{allow} = 127\ in^3 (24\ ksi) = 3048\ kip \cdot in$$

Table 13-1, Case 4

$$M_{MAX} = \frac{wL^2}{8} = M_{allow}$$

$$w = \frac{8\,M_{allow}}{L^2} = \frac{8(3048\ kip \cdot in)}{(240\ in)^2} = 0.423\ kip/in$$

$$w = 0.423\ kip/in \times \frac{12\ in}{ft} = 5.08\ kip/ft$$

Deflection

Table 16-1, case 7

$$\delta_{MAX} = \frac{5wL^4}{384\,EI} = \delta_{allow}$$

$$w = \frac{384\,EI\,\delta_{allow}}{5\,L^4}$$

$$= \frac{384(30\,000\ \text{kip/in}^2)(1330\ \text{in.}^4)(0.667\ \text{in})}{5\,(240\ \text{in})^4}$$

$$= \frac{10{,}219{,}507{,}200\ \text{kip}\cdot\text{in}^3}{16{,}588{,}800{,}000\ \text{in}^4}$$

$$= 0.616\ \text{kip/in} \times \frac{12\ \text{in}}{\text{ft}}$$

$$= 7.393\ \text{kip/ft}$$

Step 3.

$w_{MAX} = 5.08$ kip/ft (Flexural Stress σ_{MAX})

$w_{MAX} = 7.393$ kip/ft (Deflection δ_{MAX})

$\therefore$ Flexural Stress governs

and $w_{MAX} = 5.08$ kip/ft

Step 5. Check Shear for $w_{MAX} = 5.08$ kip/ft

Table 13-1, Case 4

$$V_{MAX} = \frac{wL}{2} = \frac{(5.08\ \text{kip/ft})(20\ \text{ft})}{2} = 50.8\ \text{kip}$$

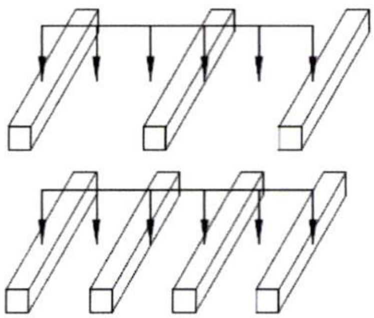
$$\tau_{avg} = \frac{V_{MAX}}{d\,t_w} = \frac{(50.8\ \text{kip})}{(20.99\ \text{in})(0.4\ \text{in})} = 6.05\ \text{ksi} < \tau_{allow} = 14.5\ \text{ksi}$$

✓ OK for shear

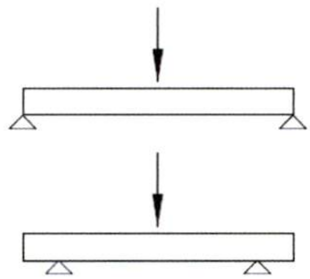
Strategies to Reduce Deflection in Beams
https://perchpro.com/2017/10/01/monthly-mechanics-5-ways-to-reduce-deflection/
written by: Scott

1. **Decrease the load.** Obviously if you need to support a 5-ton elephant, you can't say "well, let's pretend the elephant only weighs 3 tons." But maybe you can use two beams instead of one, or reduce the spacing of floor joists, to decrease the tributary load (link) on each beam.

2. **Shorten the span**. This isn't always possible. In my project, the beams needed to span between existing pile cap supports buried in the ground, so the beam length was predetermined. But deflection is proportional to span cubed, so a small reduction in length can make a big difference.



1. DECREASE LOAD

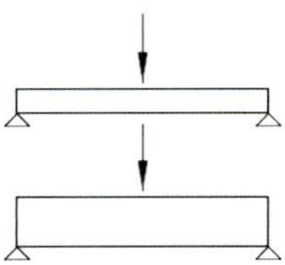


2. SHORTEN SPAN

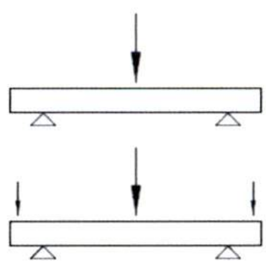
3. **Stiffen the beam.** Both the material and the shape contribute to stiffness. Steel deflects less than wood, and LVLs deflect less than dimensional lumber. Given two beams of the same material, larger cross sections with a larger moment of inertia deflect less.

4. **Add weight to the beam ends.** If you can make your beam curve down over the ends then it won't curve up as much in the middle. The problem with this strategy is that it increases the total load, which might become a structural issue for your supports.

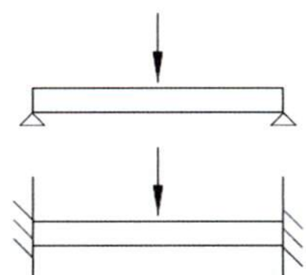
5. **Fix the supports.** To an engineer, "fix" means "prevent all movement and rotation," not "repair." Fixed-end beams deflect much less than simply supported beams. On the other hand, they transfer flexure forces to the supports as well as compression, and the supports must be beefier (to use the technical term) so they can handle the flexure.



3. INCREASE STIFFNESS



4. ADD WEIGHT TO BEAM ENDS



5. FIX BEAM ENDS

I used a combination of strategies 1, 3, and 5 to reduce deflection on my project to a level manageable for the tower crane. My client reports no visible movement while the crane is operating – that's what we want!

BY R. BRUCE HOADLEY

Simple wood beams

Beams, defined as elongated members that are loaded perpendicular to their long axis, are critical to the structure of a house. The classic example of a double or triple 2x beam supporting floor joists usually comes to mind, but joists, roof rafters, headers over windows and doors, and stair stringers are all examples of beams.

Today, builders often rely on engineered structural lumber—LVLs, PSLs, I-joists, and others—but dimensional lumber is still used widely as well. In practice, builders have no say over the strength of the wood itself; we are simply charged with using the inherent strength effectively.

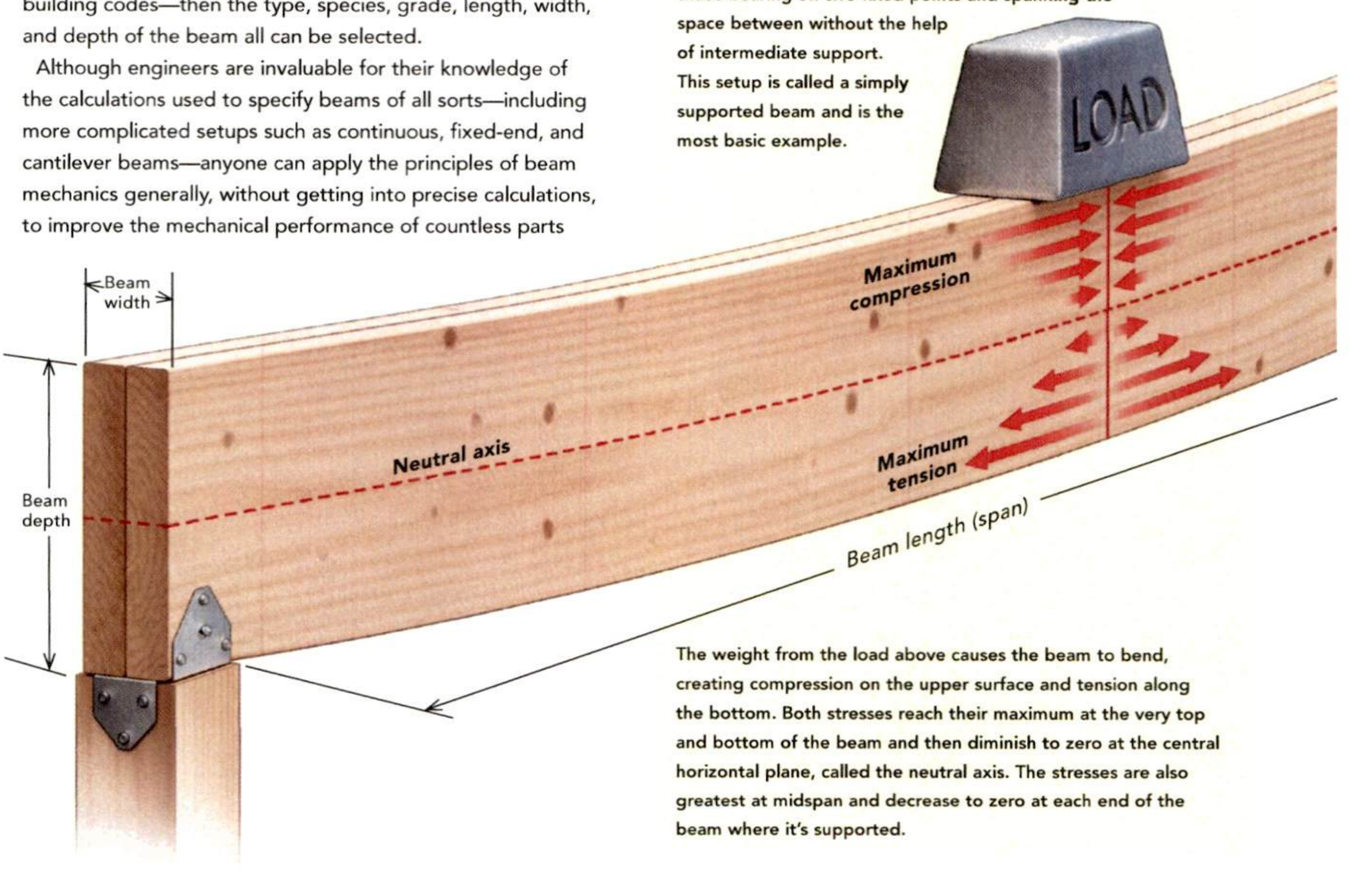
If you know the limit of acceptable deflection and how much weight a beam needs to carry—both of which are provided by building codes—then the type, species, grade, length, width, and depth of the beam all can be selected.

Although engineers are invaluable for their knowledge of the calculations used to specify beams of all sorts—including more complicated setups such as continuous, fixed-end, and cantilever beams—anyone can apply the principles of beam mechanics generally, without getting into precise calculations, to improve the mechanical performance of countless parts of a house. The decision of where to place a support column or partition wall, when to choose double 2x8s as opposed to a single 2x10, and the most effective method for stiffening an undersize joist can benefit from a basic understanding of the relationship between a beam's carrying capacity and its stiffness. Here's how it works.

This text has been adapted from the revised edition of Understanding Wood *by R. Bruce Hoadley (2000), available as an ebook at tauntonstore.com.*

The fundamentals

Let's consider a center-loaded dimensional-lumber beam that's bearing on two fixed points and spanning the space between without the help of intermediate support. This setup is called a simply supported beam and is the most basic example.



The weight from the load above causes the beam to bend, creating compression on the upper surface and tension along the bottom. Both stresses reach their maximum at the very top and bottom of the beam and then diminish to zero at the central horizontal plane, called the neutral axis. The stresses are also greatest at midspan and decrease to zero at each end of the beam where it's supported.

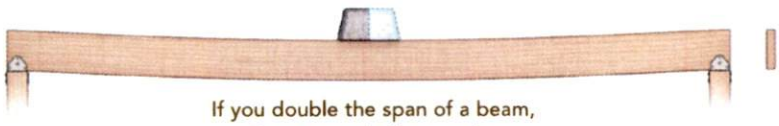
Drawings: Christopher Mills

KNOW YOUR BEAM OPTIONS

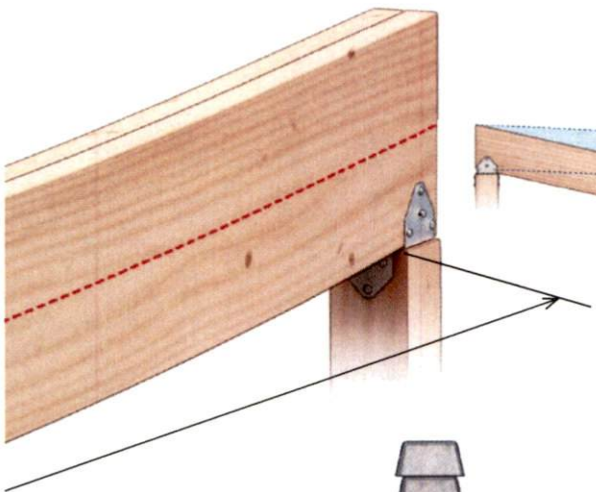
Builders face two primary considerations when choosing a beam: first, how much it can carry and what factors influence its carrying capacity; second, how much it will deflect and what factors influence its deflection.

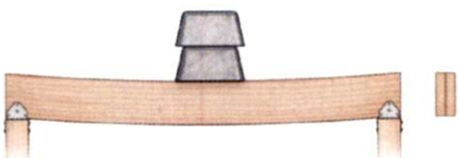
The grade and species of a beam have an effect in this regard. For example, a wood species that is twice as strong can carry twice as much weight, and a species with twice the bending tolerance—known as modulus of elasticity—will deflect half as much. However, this information is useful only if you have lots of wood species to choose from. Framing lumber is typically offered in just a few species, so the more useful information is likely to be how changes to the length, width, and depth of a beam will affect its carrying capacity and deflection. These changes are either direct (increase X, and Y increases), or inverse (increase X, and Y decreases).

Load-bearing capacity

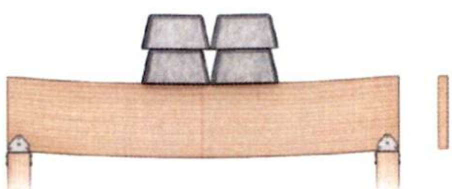


If you double the span of a beam, it can carry half as much.



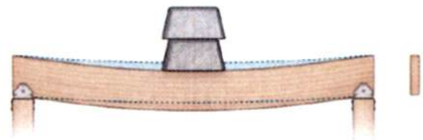


If you double the width of a beam, it can carry twice as much.

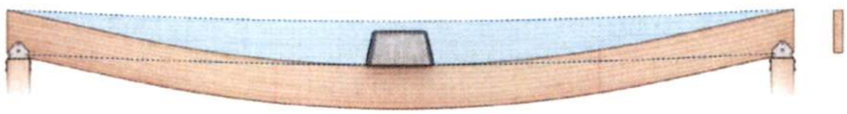


If you double the depth of a beam, it can carry four times as much.

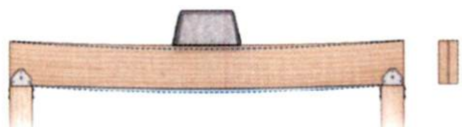
The amount of deflection



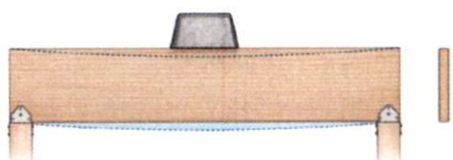
If you double the load on a beam, it will deflect twice as much.



If you double the span of a beam, it will deflect eight times as much.



If you double the width of a beam, it will deflect half as much.



If you double the depth of a beam, it will deflect ⅛ its original amount.